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On generalized vorticity-conservation laws

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We present here a new type of generalization of the conservation law of circulation, in two dimensions, and of the helicity-conservation law in three dimensions assuming zero potential vorticity. The conserved quantities keep their usual expressions $\int\int \boldsymbol{k}\cdot \mathrm{rot}\, \boldsymbol{v}\, \mathrm{d}_2 \boldsymbol{r}$, $\int\int\int \boldsymbol{v}\cdot \mathrm{rot}\, \boldsymbol{v}\, \mathrm{d}_3 \boldsymbol{r}$, respectively, as in the barotropic case.

These generalizations are based upon the general formulation of conservation laws in terms of potentials (here the Clebsch potentials). The conserved currents that we derive are expressible in terms of the ordinary physical variables $(\boldsymbol{r}, t, \boldsymbol{v}, P, \rho)$ only, as they should.

1. Introduction

One of the fundamental theorems of fluid dynamics states that the velocity circulation along a closed contour moving with the fluid is a constant (Kelvin 1868) if the flow is assumed barotropic. In three dimensions it has more recently been shown (Moreau 1961; Betchov 1961; Moffatt 1969) that the quantity

$$H = \int\int\int \boldsymbol{v}\cdot \mathrm{rot}\, \boldsymbol{v}\, \mathrm{d}_3 \boldsymbol{r},$$

called *helicity*, is a constant, still assuming isentropic flow. Several authors have attempted to remove that restriction, and have looked for generalizations also valid for non-barotropic flow. Eckart (1960) has shown that Kelvin's theorem still holds if one replaces the circulation of velocity $\boldsymbol{v}$ by that of $\boldsymbol{v}-\eta\,\boldsymbol{\nabla} S$, where η is one of the potentials occurring in the Clebsch transformation. (Hereinafter we refer to these as the Clebsch potentials, for brevity.) Mobbs (1981) has developed this idea and has given corresponding generalizations of the Helmholtz theorems, and of the helicity-conservation law as well.

In spite of the interest of such works, it must be pointed out that Mobbs' conservation laws are not expressible in terms of the usual physical variables only $(\boldsymbol{r}, t, P, \boldsymbol{v}, \rho)$, and involve 'potentials'. Such generalizations are in any case far from unique.

We present here a different type of generalization, of the two-dimensional velocity-circulation law, and of the three-dimensional helicity-conservation law assuming zero potential vorticity. The conserved quantities still read

$$\int\int \boldsymbol{k}\cdot \mathrm{rot}\, \boldsymbol{v}\, \mathrm{d}_2 \boldsymbol{r}, \quad \int\int\int \boldsymbol{v}\cdot \mathrm{rot}\, \boldsymbol{v}\, \mathrm{d}_3 \boldsymbol{r}$$

respectively, as in the barotropic case; their expression is given in terms of well-defined physical variables only, and does not involve potentials.

These two generalizations are most conveniently derived through the *Clebsch transformation* for the velocity field.

We first recall some basic results underlying the Clebsch transformation (Clebsch 1859; Lamb 1932; Seliger & Whitham 1968) and the general formulation of conservation laws; we then proceed with the explicit derivation of the two proposed generalizations, assuming adiabatic flow and an arbitrary entropy distribution.

2. The Clebsch transformation

The Clebsch transformation for the velocity field,

$$\boldsymbol{v} = \nabla\phi + \alpha\,\nabla\beta,$$

was first introduced during the 19th century (Clebsch 1859) and applied to barotropic flow only. Herivel (1955) proposed a similar formula applying to non-barotropic flow:

$$\boldsymbol{v} = \nabla\phi + \eta\,\nabla S,$$

but his result was not of general validity. Serrin (1959) and Lin (1963) introduced additional terms:

$$\boldsymbol{v} = \nabla\phi + \eta\,\nabla S + \sum_{i=1}^{3} \alpha_i\,\nabla\beta_i,$$

making the result completely general, at the expense of introducing six additional potentials α_i and β_i. Seliger & Whitham (1968) have indicated that for the most general flow the velocity field can be written in terms of four potentials only, instead of eight, as follows:

$$\boldsymbol{v} = \nabla\phi + \eta\,\nabla S + \alpha\,\nabla\beta, \tag{1}$$

where ϕ, η, α and β are four potentials whose evolution is governed by the equations

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \frac{\mathrm{d}\beta}{\mathrm{d}t} = 0, \quad \frac{\mathrm{d}\eta}{\mathrm{d}t} = T, \quad \frac{\mathrm{d}\phi}{\mathrm{d}t} = \tfrac{1}{2}v^2 - w, \tag{2}$$

where w is the specific enthalpy, defined by the thermodynamical relation

$$\mathrm{d}w = T\,\mathrm{d}S + \frac{\mathrm{d}P}{\rho}, \tag{3}$$

and T, P, ρ and S denote the temperature, pressure, density and entropy; the operator $\mathrm{d}/\mathrm{d}t \equiv \partial/\partial t + \boldsymbol{v}\cdot\nabla$. Thus α and β are Lagrangian coordinates, as well as the entropy S:

$$\mathrm{d}S/\mathrm{d}t = 0. \tag{4}$$

They are in general *multivalued* functions of $\boldsymbol{r}$ and t. The vorticity is given by

$$\operatorname{rot}\boldsymbol{v} = \nabla\eta \wedge \nabla S + \nabla\alpha \wedge \nabla\beta. \tag{5}$$

The essential property of the system (1)–(4) is that the Euler equation

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + \frac{\nabla P}{\rho} = 0,$$

which we may also write as

$$\frac{\partial\boldsymbol{v}}{\partial t} = \boldsymbol{v} \wedge \operatorname{rot}\boldsymbol{v} + T\,\nabla S - \nabla(\tfrac{1}{2}v^2 + w), \tag{6}$$

is automatically satisfied; therefore, if potentials ϕ, η, α and β have been found at an instant t such that the velocity field is expressed by (1), that expression remains valid at all times if the potentials are made to evolve according to (2).

It is always possible to find four potentials satisfying the three constraints (1) on the initial velocity field. Therefore *the Seliger–Whitham formulae are applicable to the most general flow* satisfying the Euler equations. The Herivel formulae, on the other hand, presenting only two adjustable potentials (one if the flow is isentropic) cannot describe arbitrarily given initial velocity distributions.

The simplified form derived by Herivel will be sometimes considered *as a heuristic approach* in the present paper, keeping in mind its lack of generality, but we otherwise adopt here the Clebsch transformation in the form (1)–(5) proposed by Seliger & Whitham.

The complete generality of Seliger & Whitham's formulae has been disputed, mostly on the grounds that the Clebsch potentials involved may be multivalued or singular (Bretherton 1970; Mobbs 1982); this does not constitute a drawback here, where we are only interested in the local aspect of the theory. The conservation laws (14) and (22) that we derive have an expression that is independent of potentials (unlike Mobbs' results); the Seliger–Whitham transformation indicates that they hold at least in a neighbourhood of an arbitrary point, and thus they hold *everywhere*, by continuity. After they have been discovered – by means of the S–W transformation – it is in any case straightforward to verify them by elementary methods in all their generality, *without appealing to the Clebsch representation* any more; this has been done, for the sake of completeness, in Appendix B. Thus the question of the general validity of the Clebsch transformation is in any case irrelevant in the present work.

3. The general formulation of conservation laws

3.1. *Eulerian formulation*

The general form of a conservation law is, in three dimensions,

$$\operatorname{div}\boldsymbol{j}+\frac{\partial j_0}{\partial t}=0,$$

where $\boldsymbol{j}=(j_x,j_y,j_z)$ is the flux and j_0 the density of the conserved quantity Q:

$$Q=\iiint j_0\,\mathrm{d}_3\boldsymbol{r}, \tag{7}$$

and the components can be written down in the form of Jacobians involving three potentials A, B and C as follows (Appendix A; see also Hollmann 1964):

$$\left.\begin{aligned} j_x&=-\frac{\partial(A,B,C)}{\partial(y,z,t)},\quad j_y=-\frac{\partial(A,B,C)}{\partial(z,x,t)},\quad j_z=-\frac{\partial(A,B,C)}{\partial(x,y,t)},\\ j_0&=+\frac{\partial(A,B,C)}{\partial(x,y,z)}\equiv(\boldsymbol{\nabla}A,\boldsymbol{\nabla}B,\boldsymbol{\nabla}C),\end{aligned}\right\} \tag{8}$$

where $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})$ denotes the mixed product $\boldsymbol{a}\cdot\boldsymbol{b}\wedge\boldsymbol{c}$.

The 4-dimensional lines of current are well-defined geometrical objects, and are given by the differential equations

$$j_0\,\mathrm{d}\boldsymbol{r}=\boldsymbol{j}\,\mathrm{d}t$$

(in particular, when $\boldsymbol{j}=j_0\boldsymbol{v}$, the lines of current coincide with the fluid particles' trajectories).

Substituting for A, B and C the Clebsch potentials yields various interesting results.

Thus a variant of the *mass conservation law* results (Ertel 1960) upon the substitution $(A, B, C) = (S, \alpha, \beta)$; the corresponding density is

$$j_0 = \frac{\partial(S, \alpha, \beta)}{\partial(x, y, z)} \equiv \operatorname{rot} \boldsymbol{v} \cdot \boldsymbol{\nabla} S,$$

and the flux $\boldsymbol{j} = j_0 \boldsymbol{v}$ reduces to the convection current; under such conditions the quantity $e \equiv j_0/\rho$ is therefore a *Lagrangian variable*:

$$\frac{\mathrm{d}e}{\mathrm{d}t} \equiv \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\rho} \operatorname{rot} \boldsymbol{v} \cdot \boldsymbol{\nabla} S\right) = 0;$$

it is called the 'potential vorticity' (Ertel 1942; Katz & Lynden-Bell 1982). The conserved quantity $\iiint e(S, \alpha, \beta)\, \rho\, \mathrm{d}_3 \boldsymbol{r}$ is obviously related to the law of mass conservation.†

3.2. *Lagrangian formulation*

For obvious physical reasons (namely Galilean invariance), the Eulerian flux $\boldsymbol{j}$ must contain a term $j_0 \boldsymbol{v}$, which represents the 'convection term', and a residue $\boldsymbol{j}_{\mathrm{L}}$, which may be called the 'Lagrangian flux':

$$\boldsymbol{j} = j_0 \boldsymbol{v} + \boldsymbol{j}_{\mathrm{L}}.$$

It is straightforward to show that the term $\boldsymbol{j}_{\mathrm{L}}$ is given by the same formulae (8) as $\boldsymbol{j}$ itself, provided that the partial derivatives $\partial/\partial t$ are replaced by the total derivatives $\mathrm{d}/\mathrm{d}t$. The proof goes as follows.

Consider the x-component of $\boldsymbol{j}_{\mathrm{L}}$: from (8) and the above definition of $\boldsymbol{j}_{\mathrm{L}}$ we have

$$-j_{\mathrm{L}x} \equiv \frac{\partial(A, B, C)}{\partial(y, z, t)} + v_x \frac{\partial(A, B, C)}{\partial(y, z, x)}$$

$$= \frac{\partial(A, B, C)}{\partial(y, z, t)} + v_x \frac{\partial(A, B, C)}{\partial(y, z, x)} + v_y \frac{\partial(A, B, C)}{\partial(y, z, y)} + v_z \frac{\partial(A, B, C)}{\partial(y, z, z)}$$

(the last two determinants are both identically zero, having two column vectors proportional). What is written on the second line is just $\partial(A, B, C)/\partial(y, z, t)$ with $\partial/\partial t$ replaced by $\partial/\partial t + v_x\, \partial/\partial x + v_y\, \partial/\partial y + v_z\, \partial/\partial z$ as stated. This completes the proof of the announced result.

Taking account of the continuity equation, it is easily seen that the Lagrangian formulation of a conservation law reads

$$\operatorname{div} \boldsymbol{j}_{\mathrm{L}} + \rho \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{j_0}{\rho}\right) = 0. \tag{9}$$

The term $\rho_{\mathrm{L}} \equiv j_0/\rho$ may be called the Lagrangian density.

This procedure usually results in a notably simplified formalism.

† This is evident from the fact that the two conserved currents are everywhere parallel: the lines of current coincide with the particles' trajectories. It must be noted that the two conserved quantities ($\iiint \rho\, \mathrm{d}_3 \boldsymbol{r}$ and $\iiint \rho e\, \mathrm{d}_3 \boldsymbol{r}$) still differ from one another. Furthermore, we do not mean of course that Ertel's discovery of a new explicit Lagrangian variable (the potential vorticity) did not constitute a result of fundamental importance in theoretical fluid dynamics.

4. The conservation law of circulation in the case of two dimensions

Conservation laws in two dimensions assume a simpler form:

$$j_x = \frac{\partial(A,B)}{\partial(y,t)}, \quad j_y = \frac{\partial(A,B)}{\partial(t,x)}, \quad j_0 = \frac{\partial(A,B)}{\partial(x,y)}. \tag{10}$$

The above representation is sometimes used in magnetohydrodynamics for the magnetic field, which is divergence-free; the potentials A and B are then usually called the Euler potentials.

We first consider the simplified Clebsch transformation:

$$\boldsymbol{v} = \nabla\phi + \eta\,\nabla S, \quad \operatorname{rot}\boldsymbol{v} = \nabla\eta \wedge \nabla S, \tag{11}$$

which is only valid for a particular class of flows. The result that we derive *will nevertheless turn out to be completely general.*

Choosing now for A and B the potentials η and S yields a conservation law, with associated density

$$j_0 = \frac{\partial(\eta,S)}{\partial(x,y)} = \boldsymbol{k}\cdot\nabla\eta\wedge\nabla S = \boldsymbol{k}\cdot\operatorname{rot}\boldsymbol{v}, \tag{12}$$

where $\boldsymbol{k}$ is the (constant) unit vector normal to the flow; thus the conserved quantity is, by Stokes theorem,

$$C \equiv \iint \boldsymbol{k}\cdot\operatorname{rot}\boldsymbol{v}\,\mathrm{d}x\,\mathrm{d}y = \oint \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{l}; \tag{13}$$

that is, the *velocity circulation*. Taking account of the flux $\boldsymbol{j}$, the conservation law reads explicitly

$$\operatorname{div}\{(\boldsymbol{k}\cdot\operatorname{rot}\boldsymbol{v})\,\boldsymbol{v} + \boldsymbol{k}\wedge T\nabla S\} + \frac{\partial}{\partial t}(\boldsymbol{k}\cdot\operatorname{rot}\boldsymbol{v}) = 0, \tag{14}$$

in which $(\boldsymbol{k}\cdot\operatorname{rot}\boldsymbol{v})\,\boldsymbol{v}$ merely represents the convection term.

Thus in the present case the Lagrangian flux is

$$\boldsymbol{j}_{\mathrm{L}} = \boldsymbol{k}\wedge T\nabla S. \tag{15}$$

The above results (12)–(15) are independent of the restrictive assumption (11). In general, using Seliger & Whitham's formula (5), the conserved current is *a sum of two conserved currents* of the general form (10), namely

$$j_0 = \frac{\partial(\eta,S)}{\partial(x,y)} + \frac{\partial(\alpha,\beta)}{\partial(x,y)}, \tag{16}$$

from which the formulae (12)–(15) follow, without modification (see also Appendix B for another, elementary, proof).

If the flow is isentropic the Lagrangian flux (15) vanishes; the conservation law then reduces to a variant of the mass-conservation law, i.e. the Lagrangian density $(\boldsymbol{k}\cdot\operatorname{rot}\boldsymbol{v})/\rho$ becomes a Lagrangian variable, as is well known:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{k}\cdot\operatorname{rot}\boldsymbol{v}}{\rho}\right) = 0. \tag{17}$$

Under such circumstances the integral $\iint \boldsymbol{k}\cdot\operatorname{rot}\boldsymbol{v}\,\mathrm{d}x\,\mathrm{d}y$ is a constant on any finite surface moving with the fluid; that integral is just the velocity circulation, and the Kelvin theorem is thus recovered.

5. Three-dimensional flow: the helicity-conservation law

In the case of three dimensions, it has been shown by Moffatt (1969) (see also Moreau 1961; Betchov 1961) that the helicity

$$H = \iiint \boldsymbol{v}\cdot \mathrm{rot}\, \boldsymbol{v}\, \mathrm{d}_3 \boldsymbol{r} \tag{18}$$

is a constant for isentropic flow. We now show that the isentropic assumption can be relaxed, at least for the (rather large) class of flows characterized by a *vanishing potential vorticity*

$$e \equiv \frac{1}{\rho} \mathrm{rot}\, \boldsymbol{v}\cdot \nabla S = 0 \tag{19}$$

(that is, the class of flows for which (19) holds at some 'initial' instant t; it also holds at all times during the subsequent evolution, since $\mathrm{d}e/\mathrm{d}t = 0$).

We again start with the simplified Clebsch transformation (11):

$$\boldsymbol{v} = \nabla\phi + \eta\, \nabla S, \tag{20}$$

but the result will be seen to apply independently of that restricting assumption.

Let us choose for potentials $(A, B, C) = (S, \phi, \eta)$; the resulting conservation law has density

$$j_0 = \frac{\partial(S, \phi, \eta)}{\partial(x, y, z)} = \boldsymbol{v}\cdot \mathrm{rot}\, \boldsymbol{v} \tag{21}$$

and expresses the *conservation of helicity H* (given by (18)). Explicitly we obtain by means of the general formulae (8) the helicity-conservation law

$$\mathrm{div}\, \{(w - \tfrac{1}{2}v^2)\, \mathrm{rot}\, \boldsymbol{v} + \boldsymbol{v} \wedge T \nabla S\} + \rho \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{v}\cdot \mathrm{rot}\, \boldsymbol{v}}{\rho}\right) = 0, \tag{22}$$

written according to the Lagrangian formulation (9). The total flux, including the convection term, is

$$\boldsymbol{j} = (\boldsymbol{v}\cdot \mathrm{rot}\, \boldsymbol{v})\, \boldsymbol{v} + (w - \tfrac{1}{2}v^2)\, \mathrm{rot}\, \boldsymbol{v} + \boldsymbol{v} \wedge T \nabla S. \tag{23}$$

The above conservation law (22) and (23) applies independently of the restrictive assumption (20). The most direct way to see this is to observe that the conserved current is *the sum of three conserved currents* of the general form (8); namely, using Seliger & Whitham's formula (1):

$$\boldsymbol{v}\cdot \mathrm{rot}\, \boldsymbol{v} \equiv (\nabla\phi, \nabla\alpha, \nabla\beta) + (\nabla\phi, \nabla\eta, \nabla S) + (\alpha\, \nabla\eta - \eta\, \nabla\alpha, \nabla S, \nabla\beta).$$

Under the restriction (19), $(\nabla\alpha, \nabla S, \nabla\beta) = 0$, and we may rewrite

$$j_0 \equiv \boldsymbol{v}\cdot \mathrm{rot}\, \boldsymbol{v} \equiv (\nabla\phi, \nabla\alpha, \nabla\beta) + (\nabla\phi, \nabla\eta, \nabla S) + (\nabla(\alpha\eta), \nabla S, \nabla\beta), \tag{24}$$

which is of the required form.†

In the barotropic case the last two currents vanish, and one gets

$$j_0 \equiv \boldsymbol{v}\cdot \mathrm{rot}\, \boldsymbol{v} = (\nabla\phi, \nabla\alpha, \nabla\beta). \tag{25}$$

† It is of course also feasible to derive the helicity-conservation law (22) by elementary methods, without appealing to the formalism (8) and (24) in terms of potentials. This has been done for completeness in Appendix B.

The non-barotropic generalization proposed by Mobbs (1981) coincides formally with the above formula:

$$j_{0\,\mathrm{Mobbs}} = (\nabla\phi, \nabla\alpha, \nabla\beta)$$
$$= (\boldsymbol{v} - \eta\,\nabla S)\cdot(\mathrm{rot}\,\boldsymbol{v} - \nabla\eta \wedge \nabla S). \qquad (26)$$

6. Conclusion

Several authors (e.g. Eckart 1960; Mobbs 1981) have attempted to generalize the vorticity theorems and helicity-conservation law to non-barotropic flow. Eckart's results suggest the existence of an intimate relation between the Clebsch potentials and the vorticity-conservation laws.

This relation is here elucidated through the consideration of the canonical formulation of conservation laws in terms of potentials (8). That formulation seems to constitute a powerful tool for analytical studies, although it does not appear to have been frequently used until now in fluid dynamics (see, however, Hollmann 1964).

The generalized conserved currents that we derive are expressible in terms of ordinary physical variables only.

Appendix A. The canonical formulation of conservation laws

We introduce for convenience the notation $x_0 \equiv t$. Having defined (§3.1) the lines of current, it is natural to introduce four potentials x'_0, x'_1, x'_2 and x'_3 such that the lines are the intersections of surfaces

$$x'_i = \text{constant} \quad (i = 1, 2, 3),$$

and x'_0 may be kept arbitrary. Choosing the Cartesian metric in these new coordinates:

$$\mathrm{d}s^2 = \sum_{i=0}^{3} (\mathrm{d}x'_i)^2,$$

let us write J'_0, J'_1, J'_2 and J'_3 for the contravariant components of the 4-vector current. It is geometrically evident that the last three components are identically zero; then the condition that the vector be divergence-free reads

$$\sum_{i=0}^{3} \frac{\partial}{\partial x'_i} J'_i = 0, \quad \text{i.e.} \quad \frac{\partial J'_0}{\partial x'_0} = 0,$$

which integrates as $J'_0 = f(x'_1, x'_2, x'_3)$. Now let us go back to the original coordinate system by means of the well-known transformation formula for the divergence (Landau & Lifshitz 1951)

$$\sum_{i=0}^{3} \frac{\partial}{\partial x_i} \left\{ \frac{\partial(\boldsymbol{x}')}{\partial(\boldsymbol{x})} J_i \right\} = \frac{\partial(\boldsymbol{x}')}{\partial(\boldsymbol{x})} \sum_{i=0}^{3} \frac{\partial J'_i}{\partial x'_i},$$

where $\partial(\boldsymbol{x}')/\partial(\boldsymbol{x})$ is the Jacobian of the coordinate transformation, and J_i is given in terms of the J'_i by the vector-transformation formula (Landau & Lifshitz 1951)

$$J_i = \sum_{k=0}^{3} \frac{\partial x_i}{\partial x'_k} J'_k.$$

In the present case the only contribution comes from the $k = 0$ term, hence

$$J_i = \frac{\partial x_i}{\partial x'_0} f(x'_1, x'_2, x'_3).$$

Let us redefine

$$j_i = \frac{\partial(\boldsymbol{x}')}{\partial(\boldsymbol{x})} J_i \quad (i = 0, 1, 2, 3),$$

in order that the conservation law read

$$\sum_{i=0}^{3} \frac{\partial j_i}{\partial x_i} = 0$$

as usual. Then we have, by an elementary property of Jacobians,

$$j_i = f(x_1', x_2', x_3') \frac{\partial(x_i, x_1', x_2', x_3')}{\partial(x_0, x_1, x_2, x_3)}.$$

That is,

$$j_0 = f(x_1', x_2', x_3') \frac{\partial(x_1', x_2', x_3')}{\partial(x_1, x_2, x_3)}, \quad \text{etc.}$$

It is always possible to choose new potentials A, B and C such that the Jacobian $\partial(A, B, C)/\partial(x_1', x_2', x_3')$ be equal to an arbitrarily given function $f(x_1', x_2', x_3')$, since that amounts to a single partial differential equation for the three unknown functions A, B and C. Therefore we obtain the following *canonical form of the most general conservation law*, in terms of three potentials:

$$j_0 = \frac{\partial(A, B, C)}{\partial(x_1, x_2, x_3)}, \quad \text{etc.},$$

as in (8).

The potentials may be multivalued, even singular, but that is of no consequence whatsoever *as long as the current components j_i remain expressible in terms of well-defined physical variables only* (such as $\boldsymbol{r}, t, P, \boldsymbol{v}$ and S).

In general, what is called here the conserved quantity

$$Q \equiv \iiint j_0 \, \mathrm{d}_3 \boldsymbol{r}$$

need not always be a strict constant, if the flux of $\boldsymbol{j}$ at infinity does not vanish. What matters essentially is that, in an arbitrary volume V bounded by a surface S the variation of $Q_V \equiv \iiint_V j_0 \, \mathrm{d}_3 \boldsymbol{r}$ is determined by the flux $\iint_S \boldsymbol{j} \cdot \mathrm{d}_2 \boldsymbol{S}$. Then the conservation law *exactly determines Q_V at all times* when the initial conditions and the evolution of the physical conditions at the boundary S are given; it thus plays the role of a first integral of the equations.

Appendix B. Elementary proofs of the new conservation laws

It is of course possible to give proofs of the conservation laws (14) and (22), without mentioning the Clebsch potentials.

Equation (14) may be derived by means of Vazsonyi's equation as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\operatorname{rot} \boldsymbol{v}}{\rho}\right) = \frac{1}{\rho}(\operatorname{rot} \boldsymbol{v} \cdot \boldsymbol{\nabla}) \boldsymbol{v} + \frac{1}{\rho} \boldsymbol{\nabla} T \wedge \boldsymbol{\nabla} S,$$

which is valid in three dimensions. In two dimensions Vazsonyi's equation reduces to

$$\rho \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{k} \cdot \operatorname{rot} \boldsymbol{v}}{\rho}\right) = \boldsymbol{k} \cdot \boldsymbol{\nabla} T \wedge \boldsymbol{\nabla} S$$
$$= -\operatorname{div}(\boldsymbol{k} \wedge T \boldsymbol{\nabla} S),$$

which is the Lagrangian form of (14). It is also worth mentioning that (17) follows directly from Vazsonyi's equation.

Equation (22) may be derived starting from Mobbs' (1981) equation (28):

$$\rho \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{v}\cdot \operatorname{rot} \boldsymbol{v}}{\rho}\right) = (T\boldsymbol{\nabla}S - \boldsymbol{\nabla}w)\cdot \operatorname{rot}\boldsymbol{v} + \operatorname{rot}\boldsymbol{v}\cdot\boldsymbol{\nabla}(\tfrac{1}{2}v^2) + (\boldsymbol{v}, \boldsymbol{\nabla}T, \boldsymbol{\nabla}S).$$

It is straightforward to check that, for zero potential vorticity, the right-hand side reduces to

$$-\operatorname{div}\{(w-\tfrac{1}{2}v^2)\operatorname{rot}\boldsymbol{v} + \boldsymbol{v}\wedge T\boldsymbol{\nabla}S\},$$

which proves (22).

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